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Note on the symmetries of certain material tensors for a particle in Stokes flow

By E. J. HINCH
Department of Applied Mathematics and Theoretical Physics, University of Cambridge

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Consider a single rigid particle of arbitrary shape moving in a viscous fluid. A given point O of the body has velocity $\mathbf{u}$ and the angular velocity of the body is $\boldsymbol{\omega}$. The undisturbed flow far from the particle is a uniform velocity $\mathbf{U}$ together with a general linear flow composed of a vorticity $2\boldsymbol{\Omega}$ and a pure strain $\mathbf{E}$. The particle exerts on the fluid a force $\mathbf{F}$, and a couple $\mathbf{L}$ and a stresslet $\mathbf{S}$ specified relative to O. The stresslet is a second-order tensor defined by Batchelor (1970, p. 562) as the symmetric part of the force dipole strength for the particle and is the particle contribution to the symmetric part of the bulk stress tensor. It should be noted that the values of the translational velocity, the couple and the stresslet will depend on the arbitrary choice of the point O of the body.

Let the motion be sufficiently slow for the creeping flow equations to be obeyed (low Reynolds numbers). Then from the linearity of the governing equations and boundary conditions $\mathbf{F}$, $\mathbf{L}$ and $\mathbf{S}$ must each be linear in $\mathbf{U}-\mathbf{u}$, $\boldsymbol{\Omega}-\boldsymbol{\omega}$ and $\mathbf{E}$. At first sight an expression of these relationships might appear to require nine different material tensors of various ranks which all depend on the particle's shape orientation and the position of the point O:

$$\begin{pmatrix}\mathbf{F}\\ \mathbf{L}\\ \mathbf{S}\end{pmatrix} = \mu\begin{pmatrix}\mathbf{A} & \mathbf{P}' & \mathbf{Q}'\\ \mathbf{P}'' & \mathbf{B} & \mathbf{R}'\\ \mathbf{Q}'' & \mathbf{R}'' & \mathbf{C}\end{pmatrix}.\begin{pmatrix}\mathbf{U}-\mathbf{u}\\ \boldsymbol{\Omega}-\boldsymbol{\omega}\\ \mathbf{E}\end{pmatrix}.$$

We shall see, however, that there are symmetries such that the number of independent tensors reduces to six. By definition $\mathbf{Q}''$, $\mathbf{R}''$ and $\mathbf{C}$ are symmetric in their first pair of suffices, as are $\mathbf{Q}'$, $\mathbf{R}'$ and $\mathbf{C}$ in their last pair.

Now Lorentz (1896) showed that, for two solutions of the slow viscous equations of motion (velocity fields $\mathbf{u}'(\mathbf{x})$ and $\mathbf{u}''(\mathbf{x})$ with associated stresses $\boldsymbol{\sigma}'$ and $\boldsymbol{\sigma}''$) in a volume bounded by the same surface S with unit normal $\mathbf{n}$,

$$\int_s \mathbf{u}'.\boldsymbol{\sigma}''.\mathbf{n}dS = \int_s \mathbf{u}''.\boldsymbol{\sigma}'.\mathbf{n}dS,$$

by using the divergence theorem and the Newtonian form of the stress. This generalized reciprocal theorem yields (see below) six non-trivial symmetries

$$A_{ij} = A_{ji}, \quad B_{ij} = B_{ji}, \quad P''_{ij} = P'_{ji};$$
$$Q''_{ijk} = Q'_{kij}, \quad R''_{ijk} = R'_{kij}, \quad C_{ijkl} = C_{klij}.$$

The first three are known (e.g. see Brenner 1963, 1964), but the remaining three are not and were recently overlooked by Batchelor (1970) and by Brenner (1972). The symmetry of the matrix of all the material tensors which relates the generalized fluxes to the conjugate forces is reminiscent of Onsager's principle, although only mechanical and no thermodynamic arguments are necessary in this problem.

The symmetries have powerful and unexpected consequences. For example, the relation between $\mathbf{R}'$ and $\mathbf{R}''$ implies that the symmetric part of the particle contribution to the bulk stress, $\mathbf{S}$, due to a relative rotation $\boldsymbol{\Omega}-\boldsymbol{\omega}$ can be found from the couple exerted on the particle, $-\mathbf{L}$, when it is placed in a pure straining motion $\mathbf{E}$.

There are further symmetries involving quadratic and high order integral moments of the stress exerted by the particle on the fluid with the corresponding quadratic and high order forms of the undisturbed flow far from the particle, but these have a less obvious physical interpretation.

Proof of the relationship between R′ and R″

The proofs of the other five symmetries follow the same lines.

First, it is necessary to establish two hydrodynamic problems to which the reciprocal theorem can be applied. The two problems concern the couple $\mathbf{L}$ due to the straining motion $\mathbf{E}$ and the stresslet $\mathbf{S}$ due to the relative rotation $\boldsymbol{\Omega}-\boldsymbol{\omega}$, for $\mathbf{R}'$ and $\mathbf{R}''$ respectively. To avoid difficulties at infinity, the velocity fields $\mathbf{u}'$ and $\mathbf{u}''$ are chosen as the disturbances caused by the presence of the particle, which vanish far from the particle. The remaining boundary conditions on the slow viscous flows are then

$$\mathbf{E}.\mathbf{x}+\mathbf{u}'(\mathbf{x}) = 0,$$

$$\boldsymbol{\Omega}\times\mathbf{x}+\mathbf{u}''(\mathbf{x}) = \boldsymbol{\omega}\times\mathbf{x}$$

on the surface of the particle (A). The definitions of the material tensor $\mathbf{R}'$ and of the couple $\mathbf{L}$ exerted by the particle on the fluid are

$$\mathbf{R}':\mathbf{E} = \mathbf{L} = \int_A \mathbf{x}\times\boldsymbol{\sigma}'.\mathbf{n}\,dS.$$

Note that the undisturbed straining motion does not contribute to the couple. Similarly for $\mathbf{R}''$ and the stresslet $\mathbf{S}$,

$$\mathbf{R}''.(\boldsymbol{\Omega}-\boldsymbol{\omega}) = \mathbf{S} = \frac{1}{2}\int_A [\mathbf{x}\boldsymbol{\sigma}''.\mathbf{n}\,dS+\boldsymbol{\sigma}''.\mathbf{n}\mathbf{x}\,dS].$$

Now the reciprocal theorem involves the total bounding surface S composed of the surface of the particle A and a sphere at infinity. The contributions to the integrals from the sphere at infinity vanish because far from the particle $\mathbf{u}'$, $\mathbf{u}'' = O(r^{-1})$ and $\boldsymbol{\sigma}'$, $\boldsymbol{\sigma}'' = O(r^{-2})$. Using the preceding equations, the contributions from the surface of the particle are manipulated as follows.

$$-\int_A \mathbf{u}'.\boldsymbol{\sigma}''.\mathbf{n}\,dS = \int_A (\mathbf{E}.\mathbf{x}).\boldsymbol{\sigma}''.\mathbf{n}\,dS$$

$$= \mathbf{E}:\int_A \mathbf{x}\boldsymbol{\sigma}''.\mathbf{n}\,dS = \mathbf{E}:\mathbf{S} = \mathbf{E}:\mathbf{R}''.(\boldsymbol{\Omega}-\boldsymbol{\omega}).$$

Similarly
$$-\int_A \mathbf{u}''.\boldsymbol{\sigma}'.\mathbf{n}\,dS = (\boldsymbol{\Omega}-\boldsymbol{\omega}).\mathbf{R}':\mathbf{E}.$$

Thus the reciprocal theorem implies
$$\mathbf{E}:\mathbf{R}''.(\boldsymbol{\Omega}-\boldsymbol{\omega}) = (\boldsymbol{\Omega}-\boldsymbol{\omega}).\mathbf{R}':\mathbf{E}.$$

But the choice of $\mathbf{E}$ and $(\boldsymbol{\Omega}-\boldsymbol{\omega})$ is arbitrary. Hence we have the result
$$R''_{ijk} = R'_{kij},$$
because by definition R'' is symmetric in its first pair of suffices as is R' in its last pair.

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